Teacher notes

Unit E

The decay constant as a probability per unit time

We know that if we start with N_0 nuclei of a radioactive substance, after time Δt we will be left with $N = N_0 e^{-\lambda\Delta t}$ nuclei that will not have decayed. This means that the number of nuclei that did decay in time Δt is $\Delta N = N_0 - N = N_0(1-e^{-\lambda\Delta t})$.

If $\lambda\Delta t << 1$ we may expand

$$1-e^{-\lambda\Delta t} \approx 1-(1-\lambda\Delta t) = \lambda\Delta t$$

and so $\Delta N \approx N_0\lambda\Delta t$

The probability *p* of decay is then

$$p = \frac{\Delta N}{N_0} \approx \frac{N_0\lambda\Delta t}{N_0} = \lambda\Delta t$$

and so the probability of decay per unit time is $\frac{p}{\Delta t} = \lambda$.

So suppose that $\lambda = 5.00\ \text{min}^{-1}$. What is the probability of decay within 1 minute? Here, $\lambda\Delta t = 5.00 \times 1.0 = 5.00$ which is not less than 1, so the discussion above makes no sense; the probability is not 5! But we can express $\lambda = 5.00\ \text{min}^{-1} = \frac{5.00}{60} = \frac{1}{12}\ \text{s}^{-1}$ and ask for the probability of decay within 1 second. Now $\lambda\Delta t = \frac{1}{12} \times 1.0 = \frac{1}{12}$ and is smaller than 1. The probability of no decay is then $1-\frac{1}{12}$. The probability of no decay in 60 seconds is $(1-\frac{1}{12})^{60}$ and so the probability of decay is

$$1-(1-\frac{1}{12})^{60} = 0.99460$$

How good is this approximation? The exact number of nuclei that decayed in 1 minute is

$$N_0 - N = N_0(1-e^{-\lambda\Delta t}) = N_0(1-e^{-5})$$

and so the probability of decay is

$$\frac{N_0(1-e^{-5})}{N_0}=1-e^{-5}=0.99326$$

We have agreement to 2 decimal places.

We can make the approximation better by expressing $\lambda=\frac{1}{12\times 10^3}\,\text{ms}^{-1}$. Then repeating the above steps, the probability of decay within one minute is

$$1-(1-\frac{1}{12\times 10^3})^{60\times 10^3}=0.99326$$

which agrees with the exact answer to 5 decimal places.

So the statement "λ is the probability of decay per unit time" makes sense only if the unit of time Δt is chosen such that $\lambda\Delta t \ll 1$.

For the mathematicians: we have been given $\lambda=5.00\,\text{min}^{-1}$. We express this in a unit smaller than 1 min. Say, $\frac{5}{N}\,\text{u}^{-1}$ for some arbitrary time unit u where 1 min = N u and N is a large number. Then the probability of decay within one minute is

$$1-(1-\frac{5}{N})^N$$

As N gets larger and larger we know that $\lim_{N\to\infty}(1-\frac{5}{N})^N=e^{-5}$ and the approximate expression now becomes the exact expression $1-e^{-5}$.